OCR
Oxford Cambridge and RSA

Monday 27 June 2016 – Morning

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. If additional space is required, you should use the lined page(s) at the end of the Printed Answer Book. The question number(s) must be clearly shown.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **20** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

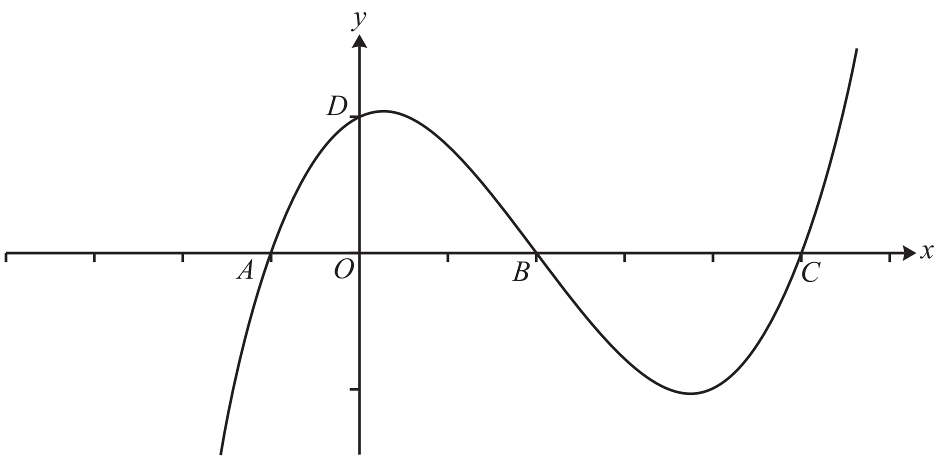
Answer **all** the questions.

1 **(i)** By first expanding $(e^x + e^{-x})^3$, or otherwise, show that $\cosh 3x \equiv 4\cosh^3 x - 3\cosh x$. **[4]**

(ii) Solve the equation $\cosh 3x = 6\cosh x$, giving your answers in exact logarithmic form. **[5]**

2 It is given that $f(x) = \dfrac{x(x-1)}{(x+1)(x^2+1)}$. Express $f(x)$ in partial fractions and hence find the exact value of $\int_0^1 f(x)\,dx$. **[6]**

3 The diagram shows the curve $y = f(x)$. Points A, B, C and D on the curve have coordinates $(-1, 0)$, $(2, 0)$, $(5, 0)$ and $(0, 2)$ respectively.



On the copy of this diagram in the Printed Answer Book, sketch the curve $y^2 = f(x)$, giving the coordinates of the points where the curve crosses the axes. **[5]**

4 You are given the equation $(2x-1)^2 - e^x = 0$.

(i) Verify that 0 is a root of the equation. **[1]**

There are also two other roots, α and β, where $0 < \alpha < \beta$.

(ii) The iterative formula $x_{r+1} = \ln(2x_r - 1)^2$ is to be used to find a root of the equation.

(a) Sketch the line $y = x$ and the curve $y = \ln(2x-1)^2$ on the same axes, showing the roots 0, α and β. **[3]**

(b) By drawing a 'staircase' diagram on your sketch, starting with a value of x that is between α and β, show that this iteration does not converge to α. **[1]**

(c) Using this iterative formula with $x_1 = 3.75$, find the value of β correct to 3 decimal places. **[3]**

(iii) Using the Newton-Raphson method with $x_1 = 1.6$, find the root α of the equation $(2x-1)^2 - e^x = 0$ correct to 5 significant figures. Show the result of each iteration. **[4]**

5 It is given that $y = \tan^{-1} 2x$.

(i) Find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 = 0$. **[3]**

(ii) Find the Maclaurin series for y up to and including the term in x^3. Show all your working. **[4]**

(iii) The result in part **(ii)**, together with the value $x = \frac{1}{2}$, is used to find an estimate for π. Show that this estimate is only correct to 1 significant figure. **[2]**

6 The equation of a curve in polar coordinates is $r = \sin 5\theta$ for $0 \leqslant \theta \leqslant \frac{1}{5}\pi$.

(i) Sketch the curve and write down the equations of the tangents at the pole. **[4]**

(ii) The line of symmetry meets the curve at the pole and at one other point A. Find the equation of the line of symmetry and the cartesian coordinates of A. **[2]**

(iii) Find the area of the region enclosed by this curve. **[4]**

7 **(i)** By using a set of rectangles of unit width to approximate an area under the curve $y = \frac{1}{x}$, show that $\sum_{x=1}^{\infty} \frac{1}{x}$ is infinite. **[4]**

(ii) By using a set of rectangles of unit width to approximate an area under the curve $y = \frac{1}{x^2}$, find an upper limit for the series $\sum_{x=1}^{\infty} \frac{1}{x^2}$. **[5]**

8 It is given that $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx$ where n is a positive integer.

(i) By writing $\sec^n x = \sec^{n-2} x \sec^2 x$, or otherwise, show that

$$(n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2} \text{ for } n > 1.$$ **[5]**

(ii) Show that $I_8 = \frac{96}{35}$. **[3]**

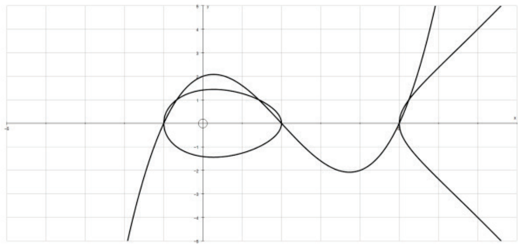
(iii) Prove by induction that I_{2n} is rational for all values of $n > 1$. **[4]**

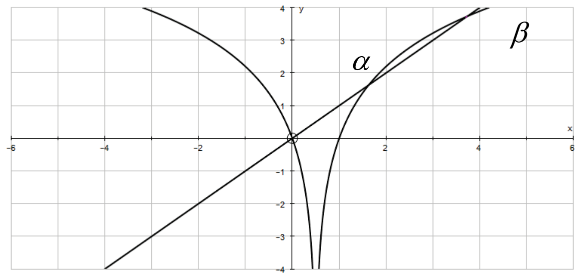
END OF QUESTION PAPER

Question			Answer	Mark	Guidance	
1	(i)		$\left(e^x+e^{-x}\right)^3=e^{3x}+3e^x+3e^{-x}+e^{-3x}$	M1	Doing the expansion	
			$=\left(e^{3x}+e^{-3x}\right)+3\left(e^x+e^{-x}\right)$	A1		
			$\Rightarrow\left(2\cosh x\right)^3=2\cosh 3x+6\cosh x$	M1	Relating cosh3x to exponentials correctly	
			$\Rightarrow 8\cosh^3 x=2\cosh 3x+6\cosh x$	A1		
			$\Rightarrow \cosh 3x=4\cosh^3 x-3\cosh x$	[4]		
	(ii)		$\Rightarrow \cosh 3x=4\cosh^3 x-3\cosh x=6\cosh x$	M1	Using result of (i)	
			$\Rightarrow 4\cosh^3 x=9\cosh x$			
			$\Rightarrow \cosh^2 x=\frac{9}{4}$ since $\cosh x\neq 0$	A1	At least one rejection needs to be stated. Or cosh$x$ ≥ 1	
			$\Rightarrow \cosh x=(\pm)\frac{3}{2}$ $\cosh x\neq-\frac{3}{2}$	A1		
			$\Rightarrow x=\pm\ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-1}\right)$			
			$=\pm\ln\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)$ or $\ln\left(\frac{3}{2}\pm\frac{1}{2}\sqrt{5}\right)$	A1 A1 [5]	A1 for each in exact form Deduct from 5 marks 1 mark for additional incorrect answers	

Question			Answer	Mark	Guidance	
			Alternative: $\cosh 3x = 6\cosh x \Rightarrow \frac{1}{2}\left(e^{3x}+e^{-3x}\right) = 3\left(e^{x}+e^{-x}\right)$ $\Rightarrow e^{3x} - 6e^{x} - 6e^{-x} + e^{-3x} = 0 \Rightarrow e^{6x} - 6e^{4x} - 6e^{2x} + 1 = 0$ let $y = e^{2x}$ $\Rightarrow y^3 - 6y^2 - 6y + 1 = 0 \Rightarrow (y+1)\left(y^2 - 7y + 1\right) = 0$ $\Rightarrow y = -1, \frac{7 \pm \sqrt{45}}{2}$ $e^{2x} \neq -1 \Rightarrow e^{2x} = \frac{7 \pm \sqrt{45}}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{7 \pm \sqrt{45}}{2}\right) = \frac{1}{2}\ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$	**M1** **A1** **A1** **A1** **A1**	Using exponentials Cubic in factorised form Rejection of $y = -1$ must be stated. **oe** in exact form Deduct from 5 marks 1 mark for additional incorrect answers	

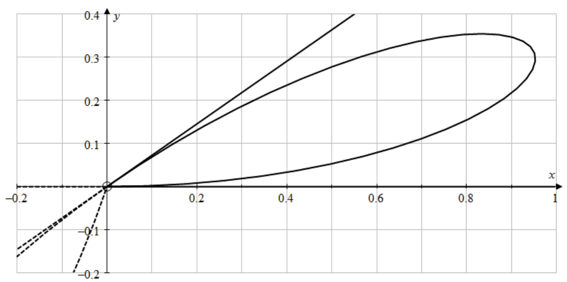
Question			Answer	Mark	Guidance	
2			$\mathrm{f}(x)=\frac{x(x-1)}{(x+1)(x^2+1)}\equiv\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+1)}$	M1	Correct partial fractions	
			$\Rightarrow A(x^2+1)+(Bx+C)(x+1)\equiv x(x-1)$ For e.g. equate coefficients $\Rightarrow A+B=1,\quad B+C=-1,\quad A+C=0$	M1	Dep on 1st M	Or sub values of x or division
			$\Rightarrow A=1, B=0, C=-1$ $\Rightarrow (\mathrm{f}(x))=\frac{1}{(x+1)}-\frac{1}{(x^2+1)}$	A1	Dep on both M marks.	
			$\Rightarrow \int_0^1 \mathrm{f}(x)\mathrm{d}x=\int_0^1\left(\frac{1}{(x+1)}-\frac{1}{(x^2+1)}\right)\mathrm{d}x$	B1	ft for integrating 1st term correctly (A/(x + 1) A ≠0)	
			$=\left[\ln(1+x)-\tan^{-1}x\right]_0^1=\ln 2-\frac{\pi}{4}$	B1	ft for subsequent term(s) correctly	
				B1	in exact form as dep on both previous B marks	
				[6]		

Question			Answer	Mark	Guidance	
3			 (-1, 0), (2, 0), (5, 0) $\left(0,\sqrt{2}\right),\left(0,-\sqrt{2}\right)$	B1 B1 B1 B1 B1 **[5]**	Symmetric but not for reflecting original curve Both ranges only and nothing more All parts cut x-axis at 90^0 Both parts cut original curve at $y = k$ and central part "egg shaped" k only approximately 1 All 5points given!	Approximate consistency

Question			Answer	Mark	Guidance	
4	(i)		$x = 0$ in equation satisfies as $e^0 = 1$.	B1 [1]		
	(ii)	(a)		B1 B1 B1 [3]	Asymptote between $x = 0$ and where it crosses x axis . +ve roots clear LH branch going through origin LH branch does not have to be complete	Allow one branch. SC1 $y = \left(\ln(2x-1)\right)^2$
		(b)	Staircase seen near middle root to be converging to β.	B1 [1]	Either starting point shown with vertical line from axis to curve or arrows on staircase lines	Follow through their curve where there are two positive roots
	(ii)	(c)	$x_1 = 3.75$ $x_2 = 3.743604...$ Leading to 3.733	B3 [3]	For correct answer B2 for 3.734 B1 for sight of 3.7436...	
	(iii)		$f(x) = (2x-1)^2 - e^x$ $\Rightarrow f'(x) = 4(2x-1) - e^x$ $\Rightarrow x_{r+1} = x_r - \dfrac{(2x-1)^2 - e^x}{4(2x-1) - e^x}$ $\Rightarrow x_2 = 1.629382......., \ x_3 = 1.629053$ Root = 1.6291	B1 M1 A1 A1 [4]	f'(x) correct **soi by x_2** Use of formula **soi by x_2** x_2 to 2dp or better Correct root stated to 5sf	 f(x) correct and their f'(x) At least 2 iterates shown

Question			Answer	Mark	Guidance	
5	(i)		$\frac{dy}{dx} = \frac{2}{1+4x^2}$ $\frac{d^2y}{dx^2} = -2\left(1+4x^2\right)^{-2} \times 8x = \frac{-16x}{\left(1+4x^2\right)^2} = -4x\left(\frac{dy}{dx}\right)^2$	B1 M1 A1 [3]	For first diffn Diffn again and making comparison	
	(ii)		When $x=0,\ y=0, \frac{dy}{dx}=2, \frac{d^2y}{dx^2}=0$ $\Rightarrow \frac{d^3y}{dx^3} + 4\left(\frac{dy}{dx}\right)^2 + 8x\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 0$ When $x=0, \frac{d^3y}{dx^3}+16=0$ $\Rightarrow (y) = 2x - \frac{8x^3}{3}$	B1 M1 A1 A1 [4]	Soi by final answer Differentiate the equation given For – 16 www Final answer	See below for alternative differentiation. SC4 Final formula from formula book including sight of $(2x)^3$. SC2 if final form only seen (i.e. no working) SC0 if final form only and wrong
	(iii)		$x = \frac{1}{2} \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}$ In series $x = 1 - \frac{1}{3} = \frac{2}{3}$ $\Rightarrow$ Estimate for $\pi = \frac{8}{3} = 2.666....$ which, correct to 1sf, $= 3$	B1 B1 [2]	soi For showing to 1sf $\pi = 3$ which is correct but to 2sf $\pi = 2.7$ which is not. www	

Question			Answer	Mark	Guidance	
			Alternative differentiation: $y'' = \dfrac{-16x}{(1+4x^2)^2} \Rightarrow y''' = \dfrac{-16(1+4x^2)^2 + 256(1+4x^2)x^2}{(1+4x^2)^4}$ or $y'' = -16x(1+4x^2)^{-2} \Rightarrow y''' = 256x^2(1+4x^2)^{-3} - 16(1+4x^2)^{-2}$		M1 is for two terms on top M1 is for 2 terms	

Question			Answer	Mark	Guidance	
6	(i)			B1 B1 B1 B1 [4]	Single enclosed loop in 1st quadrant tangent at origin (not necessarily seen) less than $\theta = \frac{\pi}{4}$ $\theta = \frac{\pi}{5}$ stated $\theta = 0$ stated –1 any extra "tangents" if 2 are correct	
	(ii)		$\theta = \frac{\pi}{10}$, $x = r\cos\theta = \cos\frac{\pi}{10}$ (≈ 0.951) $y = r\sin\theta = \sin\frac{\pi}{10}$ (≈ 0.309)	B1 B1 [2]	For θ. Allow cartesian $y = \left(\tan\frac{\pi}{10}\right)x$ For both *x* and *y* . Allow decimal values to 3sf or better	
	(iii)		$A = \frac{1}{2}\int_0^{\pi/5} r^2 \mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/5} \sin^2 5\theta \mathrm{d}\theta$ $= \frac{1}{4}\int_0^{\pi/5} (1-\cos 10\theta)\mathrm{d}\theta = \frac{1}{4}\left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5}$ $= \frac{1}{4}\left(\frac{\pi}{5} - \frac{1}{10}\sin 2\pi\right) = \frac{\pi}{20}$	M1 M1 A1 A1 [4]	Correct formula for area with correct limits Correct method to get integrand Dep on 2nd M Integral – ignore limits	or $A = \int_0^{\pi/10} r^2 \mathrm{d}\theta$

Question			Answer	Mark	Guidance	
7	(i)		Construct rectangles from $x = 1$ of height y to n or ∞ Sum of areas of rectangles $= \frac{1}{1} + \frac{1}{2} + \ldots = \sum_{1}^{\infty} \frac{1}{x}$ This area is bigger than the area under the curve $A = \int_{1}^{\infty} \frac{1}{x} \mathrm{d}x = [\ln x]_{1}^{\infty} = \infty$ Since *Sum* > *A*, the sum is infinite.	M1 A1 B1 A1 [4]	Or sum from 1 to n soi from diagram For sum and inequality stated or implied by diagram Integral from 1 to n +1or ∞ Conclusion	Or to n - 1 Condone comparison of areas for 1 to n and 1 to n - 1 If n- 1 above then this integral to n
	(ii)		Construct rectangles from $x = 2$ to the left of height y to n or ∞ Sum of areas of rectangles $= \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{2}^{\infty} \frac{1}{x^2}$ This area is less than the area under the curve $A = \int_{1}^{\infty} \frac{1}{x^2} \mathrm{d}x = \left[-\frac{1}{x} \right]_{1}^{\infty} = 0 - -1 = 1$ Since *Sum* < *A* , $\frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{2}^{\infty} \frac{1}{x^2} < 1$ $\Rightarrow \sum_{1}^{\infty} \frac{1}{x^2} < 1 + 1$ Upper limit = 2	M1 A1 B1 M1 A1 [5]	Rectangles must be under curve For sum and inequality stated or implied by diagram Area under curve For adding 1 to both sides, may appear earlier	May include the rectangle from x= 1 to the left

Question			Answer	Mark	Guidance	
8	(i)		$I_n = \int_0^{\pi/4} \sec^n x\,dx$			
			$u = \sec^{n-2} x \qquad dv = \sec^2 x\,dx$ $du = (n-2)\sec^{n-3} x.\sec x \tan x\,dx \qquad v = \tan x$	M1	Attempt at integration by parts with correct u and v'	
			$\Rightarrow I_n = \left[\sec^{n-2} x \tan x\right]_0^{\pi/4} - \int_0^{\pi/4} (n-2)\sec^{n-3} x.\sec x \tan x.\tan x\,dx$	A1	"uv" term must be seen	
			$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int_0^{\pi/4} \sec^{n-2} x.\tan^2 x.dx$	A1	Evaluating the first term	
			$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int_0^{\pi/4} \sec^{n-2} x\left(\sec^2 x - 1\right).dx$	M1	Dep on 1st M Splitting integral using $\tan^2 x = \sec^2 x - 1$	
			$= \left(\sqrt{2}\right)^{n-2} - (n-2)\left(I_n - I_{n-2}\right) \Rightarrow (n-1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}$	A1 [5]		
	(ii)		$I_8 = \frac{1}{7}\left(2^3 + 6I_6\right) = \frac{1}{7}\left(2^3 + 6.\frac{1}{5}\left(2^2 + 4I_4\right)\right)$	B1	Correct statement of formula seen anywhere	
			$= \frac{1}{7}\left(2^3 + 6.\frac{1}{5}\left(2^2 + 4.\frac{1}{3}\left(2 + 2I_2\right)\right)\right)$			
			$= \frac{1}{7}\left(8 + \frac{6}{5}\left(4 + \frac{16}{3}\right)\right) = \frac{1}{7}\left(8 + \frac{6}{5}.\frac{28}{3}\right)$	B1	Substitution of I_2 seen oe	
			$= \frac{1}{7}.\frac{96}{5} = \frac{96}{35}$	B1	Conclusion	
			Alternative: $I_2 = 1 \qquad I_4 = \frac{4}{3} \qquad I_6 = \frac{28}{15} \Rightarrow I_8 = \frac{96}{35}$	[3]	B1 for I_2 B1 For I_4 and I_6 B1	

Question			Answer	Mark	Guidance	
	(iii)		$I_2 = 1$ (which is rational). (Therefore I_{2n} is rational for $n = 1$). Let I_{2k} be rational for some value of k Then $I_{2(k+1)} = \frac{1}{(2k+1)}\left(\sqrt{2}^{2k} + 2kI_{2k}\right) = \frac{1}{(2k+1)}\left(2^k + 2kI_{2k}\right)$ Statement that this is rational and must include $\sqrt{2}^{2k} = 2^k$ is rational So if I_{2k} is rational then $I_{2(k+1)}$ is rational. But I_2 is rational so I_{2k} is rational for all $n = k$	**B1** **M1** **A1** **A1** **[4]**	Allow $n = 2$ For statement plus reduction formula and begin to look at the $\sqrt{2}$ term Reduction formula must be correct	 Assume true for $n = k$